Uniformly Valid Second-Order Solution for Supersonic Flow Over Cruciform Surfaces†

JAMES WALLACE* JOSEPH H. CLARKE**
Brown University

Summary

Considered is the second-order supersonic flow over a cruciform configuration consisting of two intersecting rectangular wings of high aspect ratio. The practical interest is in application to supersonic inlets, wing-body junctions and vehicle fins. The fundamental interest centers about identification and adjustment of the severe local failures of the ordinary second-order theory. For wings with discontinuous slopes, discontinuous potentials occur across the planar shock and square-root singularities in the velocities occur at the intersection of these shocks with the cruciform surfaces. The problem is simple enough so that these interesting features stand out clearly.

A second-order solution uniformly valid to first order is constructed by adjustment of the ordinary second-order solution obtained first. The uniformly valid solution has two different series representations in the thickness parameter. One is the ordinary second-order series in ascending integral powers of the thickness parameter which is valid in the interior of the vertex-centered undisturbed Mach cone, and the other is a series containing fractional powers which is valid adjacent to and upstream of this Mach cone. The uniformly valid solution gives the detailed wave structure and shows a flow regime upstream of the vertex-centered undisturbed Mach cone not predicted by the ordinary theory. The two solutions are otherwise identical. The wave structure consists of a pyramidal arrangement of planar shocks adjacent to and upstream of the above cone, followed by weaker oblique expansion fans and finally by two extremely weak shocks coincident with the vertex-centered undisturbed Mach cone. As an example of the above, detailed results are presented for the case of two intersecting wedges. Application of the techniques to other quasi-cylindrical problems is discussed.

Symbols

B = $\sqrt{M^2 - 1}$
c_p = pressure coefficient
f = second-order incremental improvement in perturbation potential divided by U
k = tangent of sweepback angle
M = free-stream Mach number
$\mathbf{n}$ = unit normal to shock surfaces
n = k/B
N = $(\gamma + 1)M^2/2B^2$
q = magnitude of velocity
S = shock surfaces
u = new variable defined by Eq. (38)
U = free-stream velocity, in x-direction
$\mathbf{V}$ = perturbation velocity vector
x,y,z = cartesian coordinates
$Y(x)$ = profile of cruciform wing near $y = 0$
$Z(x)$ = profile of cruciform wing near $z = 0$
γ = specific-heat ratio
ξ,η,ζ = cartesian coordinates of integration corresponding to x,y,z
μ = $\cot^{-1}B$; Mach angle
τ = characteristic thickness parameter
Φ = exact perturbation potential divided by U
φ = linearized perturbation potential divided by U

Subscripts

x,y,z = indicated partial differentiation
1 = effects due to isolated wing one
2 = effects due to isolated wing two
3 = mutual interference effects

Superscripts

c = complimentary solution
p = particular integral

Introduction

In this paper we consider the steady supersonic flow to second-order near the juncture of perpendicularly arranged surfaces. The surfaces can be regarded as those of a cruciform arrangement of thin symmetric, high aspect-ratio rectangular wings whose profiles are given by $y = \tau_1 Y(x)$ and $z = \tau_2 Z(x)$ in the first octant of a cartesian coordinate system. The small parameters τ_1 and τ_2 are to be considered positive for the symmetric or thickness case. The results apply providing the tip influence regions do not extend to the interference phenomena near the juncture. Then, the four quadrants are independent of each other and the supposed symmetry in the coordinates y and z is merely for convenience. The flow upstream of the cruciform is uniform and parallel to the x-axis with velocity U.

This problem is of practical interest because it is applicable to supersonic inlets, wing-fuselage junctures, and vehicle fins. Second-order solutions are particularly relevant for flight Mach numbers between the linear range and the hypersonic range.[1] The problem is of fundamental interest for a number of reasons. It represents a simple case where the local failure of ordinary second-order theory is quite severe whenever the slope of either wing has a discontinuity. Second-order theory then gives inadmissible discontinuities in potential across the planar misplaced shocks and square-root singularities in the velocities along the intersection of these shocks with the reference surfaces $y = 0$ and $z = 0$. These anomalies have no physical

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* Research Assistant, Div. of Engineering.
** Associate Professor, Div. of Engineering.

significance. Because of the applicability of a particular integral given by Landahl, Drougge, and Beane,[2] the problem is simple enough so that the interesting features stand out clearly.

For quasi-cylinders such as cruciform surfaces, the assumption that the perturbation velocity components as calculated in the linearized theory are $0(\tau)$ and that differentiation does not change the order leads us to assume that the exact perturbation potential Φ has a series representation

$$\Phi = \tau\varphi + \tau^2 f + 0(\tau^3)$$

This leads to an infinite set of inhomogeneous linear equations in which the second-order problem is the second step in the process. This process always implies that the disturbances are propagated along the free-stream characteristics; the iteration process gives successive incremental corrections to the velocity field but does not give corrections to the characteristic or wave surfaces.

It is known that in many cases more than one series representation is required to describe the complete flow field because order estimates valid near solid surfaces may not hold near wave phenomena. A specific example is Broderick's theory for axisymmetric flows. The other series representations are usually unknown a priori so that an immediate obstacle appears. Lighthill[3, 4] overcame this by noting that the usual assumed series expansion will fail at the singular surfaces associated with the linearized solution where one or more violations of the assumptions occur. The series near these surfaces is then modified by expanding both the dependent variable and one of the independent variables in a series in τ and a new variable u, and using this last transformation to insure a solution which does not grow increasingly singular. We modify the solution in the manner

$$\Phi = \tau\varphi(u,y,z) + \tau^2 f(u,y,z) + 0(\tau^3)$$

$$x = u + \tau x_1(u,y,z) + \tau^2 x_2(u,y,z) + 0(\tau)^3$$

where the second equation gives u as an implicit function of x,y,z and τ, and determine x_1 to make the linearized solution $\tau\varphi$ uniformly valid, x_2 to make the second-order solution $\tau^2 f$ uniformly valid, and so on. Thus a contribution is always retained that is ordinarily too small. Besides modifying the series solution, the transformation from the u,y,z to the x,y,z space also has the equivalent interpretation as a procedure for giving the correct location of the waves.

An alternate physical procedure for making certain ordinary first- and second-order solutions uniformly valid is given by the shift rules of Whitham[3] and Lighthill.[3, 4] Whitham's rule states[3] that the ordinary linearized theory gives a correct first approximation throughout the flow, provided that the value which it predicts for any physical quantity, at a given distance from the axis on the straight (approximate) Mach line, $x - (M^2 - 1)^{1/2} r =$ const., pointing downstream from a given point on a projectile surface, is reinterpreted as the value, at that distance from the axis, on the exact Mach line which points downstream from the said point. Lighthill's rule states[3] that the second-order flow field is made uniformly valid if distance downstream from the undisturbed nose Mach cone is always reinterpreted therein as distance downstream from the limit cone. The position of the shock to be inserted is bounded behind by the undisturbed Mach cone and ahead by the limit cone. Actually, Lighthill's rule is valid only if $x_1 \equiv 0$; this requirement is fulfilled in the cases he considers.[3] In Ref. 4, to find the shock strength in conical fields, he does account for the case $x_1 \neq 0$ in the expansion procedure.

In this paper we solve the problem of cruciform surfaces by the ordinary second-order theory and exhibit the nonuniform validity that occurs in the velocity field whenever the rectangular wings have discontinuous slopes. The procedure we then use to adjust these singularities draws on elements of both the physical and expansion procedures discussed above. For the simplest case of wedge profiles, we introduce shifts like those above to make the solution uniformly valid to first-order. Considering this result, we turn to the case of arbitrary cruciform surfaces and derive an expression which shows the effect of x_1 in making the solution uniformly valid, again to first order. Since the techniques involve adjusting the explicit ordinary second-order solution, the result can only be uniformly valid to first-order. But some information on rendering solutions uniformly valid to second-order is presented: we give an extended version of Lighthill's shift rule which is valid for quasi-cylinders when $x_1 = 0$.

Potential Equations

If Φ denotes the exact perturbation potential divided by U, the exact equation for Φ is, in the case of a perfect gas,

$$-B^2\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = M^2[(\gamma - 1)/2(2\Phi_x + \Phi_x^2 + \Phi_y^2 + \Phi_z^2)(\Phi_{xx} + \Phi_{yy} + \Phi_{zz}) + 2\Phi_x \Phi_{xx} + \Phi_x^2\Phi_{xx} + \Phi_y^2 \Phi_{yy} + \Phi_z^2 \Phi_{zz} + 2\Phi_y \Phi_z \Phi_{yz} + 2(1 + \Phi_x)\Phi_y \Phi_{xy} + 2(1 + \Phi_x)\Phi_z \Phi_{xz}] \quad (1)$$

Since the tangential velocity component or equivalently the velocity potential must be continuous at shock surfaces, one has

$$[\mathbf{n} \times \nabla\Phi] = 0 \quad \text{or} \quad [\Phi] = 0 \quad (2)$$

while the jump in the normal component of the velocity is given by

$$[\mathbf{n}\cdot\nabla\Phi] = \frac{2}{\gamma + 1}\left[\frac{a^2/U^2}{\mathbf{n}\cdot\nabla(x + \Phi)} - \mathbf{n}\cdot\nabla(x + \Phi)\right] \quad (3)$$

where

$$a^2/U^2 = 1/M^2 - (\gamma - 1)/2(2\Phi_x + \Phi_x^2 + \Phi_y^2 + \Phi_z^2) \quad (4)$$

The quantities on the right hand side of Eq. (3) are

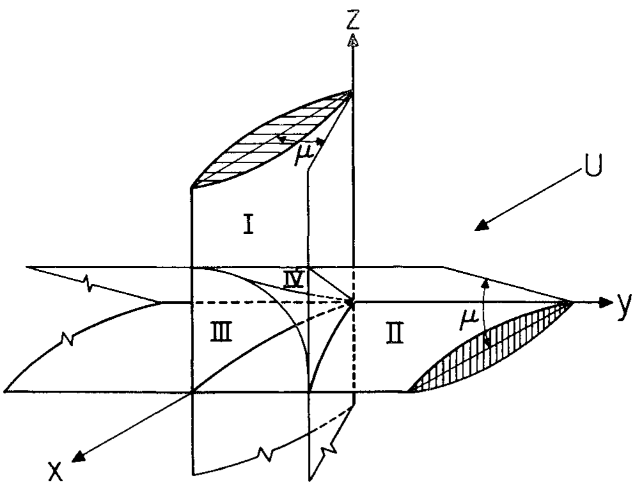

FIG. 1. Cruciform wings.

evaluated on the upstream side of the shock surfaces. In addition, the normal derivative of $U(x + \Phi)$ must vanish on solid bodies and the space derivatives of Φ must vanish upstream.

Solutions by Ordinary Second-Order Theory

Second-order theory represents the second step in an iteration procedure with respect to the parameter τ in Eq. (1). For quasi-cylindrical bodies such as cruciform wings, the representation of the potential is of the form

$$\Phi = \varphi + f + 0(\tau^3); \qquad \varphi = 0(\tau), f = 0(\tau^2) \quad (5)$$

Substitution of Eq. (5) into Eq. (1) under the assumption $\tau^2 \ll 1$, one obtains the following equations to be solved:

$$-B^2\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0 \quad (6)$$

$$-B^2 f_{xx} + f_{yy} + f_{zz} = 2M^2[(N-1)B^2\varphi_x\varphi_{xx} + \varphi_y\varphi_{xy} + \varphi_z\varphi_{xz}] \quad (7)$$

where

$$N = (\gamma + 1)M^2/2B^2 \quad (8)$$

The flow upstream of the cruciform body must be uniform with velocity U along the x axis. This boundary condition may be written

$$\varphi = f = \varphi_x = f_x = 0 \qquad \text{on } x = 0 \quad (9)$$

No explicit use in the ordinary second-order theory is made of the shock conditions Eqs. (2) and (3). The flow must be tangent to the surfaces $y = \tau_1 Y(x)$ and $z = \tau_2 Z(x)$. This gives, after expanding in a Maclaurin's series about $y = 0$ and $z = 0$ respectively, the following boundary conditions

$$\varphi_y = \tau_1 Y'(x); f_y = \tau_1 Y'(x)\varphi_x - \tau_1 Y(x)\varphi_{yy} \qquad \text{on } y = 0 \quad (10)$$

$$\varphi_z = \tau_2 Z'(x); f_z = \tau_2 Z'(x)\varphi_x - \tau_2 Z(x)\varphi_{zz} \qquad \text{on } z = 0 \quad (11)$$

The flow field contains three distinct regions: one in which the first and second-order solutions are due to the isolated wing $y = \tau_1 Y(x)$ and consequently two dimensional, one whose contributions are due to isolated wing $z = \tau_2 Z(x)$, also two dimensional, and a region where the flow of each wing interferes with the other. The region of interference is further subdivided according to whether a point is internal or external to the vertex-centered free-stream Mach cone. These regions are denoted by I, II, III, and IV respectively in Fig. 1.

The solutions to the first-order problem are

$$\varphi_1 = -(\tau_1/B)Y(x - By) \quad (12)$$

for wing 1 alone, and

$$\varphi_2 = -(\tau_2/B)Z(x - Bz) \quad (13)$$

for wing 2 alone. In the interference regions III and IV, the first-order solution is merely the sum of the above potentials, since this satisfies all boundary conditions in regions III and IV; thus

$$\varphi = \varphi_1 + \varphi_2 \quad (14)$$

indicating that the mutual interference is a second-order effect.

The solutions to the two-dimensional second-order problem are[5]

$$f_1 = -\tau_1^2 \left\{ Y(x - By)Y'(x - By) + \frac{M^2(N-1)y}{2B} \times [Y'(x - By)]^2 + \frac{M^2(N-2)}{2B^2}\int_0^{x-By}[Y'(t)]^2 dt \right\} \quad (15)$$

for wing 1 alone, and

$$f_2 = -\tau_2^2 \left\{ Z(x - Bz)Z'(x - Bz) + \frac{M^2(N-1)z}{2B} \times [Z'(x - Bz)]^2 + \frac{M^2(N-2)}{2B^2}\int_0^{x-Bz}[Z'(t)]^2 dt \right\} \quad (16)$$

for wing 2 alone. For wings with discontinuous initial slopes, these potentials are discontinuous across the surfaces $x - By = 0$ and $x - Bz = 0$. In regions III and IV, we write the second-order potential as

$$f = f_1 + f_2 + f_3 \quad (17)$$

and the problem is now to find f_3. Substitution of Eqs. (14) and (17) into Eq. (7) gives the following differential equation:

$$-B^2 f_{3xx} + f_{3yy} + f_{3zz} = 2M^2(N-1)B^2(\varphi_{1x}\varphi_{2xx} + \varphi_{2x}\varphi_{1xx}) \quad (18)$$

Solution of Eq. (18) is facilitated by introducing the particular integral given in Ref. 2

$$f_3 = f_3^c + f_3^p = f_3^c - M^2(N-1)(\varphi_1\varphi_2)_x \quad (19)$$

Using the same decomposition and particular integral in the boundary conditions Eqs. (10) and (11), we obtain the following set of equations to be solved:

$$-B^2 f_{3xx}^c + f_{3yy}^c + f_{3zz}^c = 0 \quad (20)$$

$$f_{3y}^c = \tau_1 Y'(x)\varphi_{2x} + M^2(N-1)(\varphi_1\varphi_2)_{yx} \qquad \text{on } y = 0 \text{ in III} \quad (21)$$

$$f_{3z}{}^c = \tau_2 Z'(x)\varphi_{1x} + M^2(N-1)(\varphi_1\varphi_2)_{zz} \quad \text{on } z = 0 \text{ in III} \qquad (22)$$

Since the flow in each quadrant is independent of the others, we can use the concept of reflecting surfaces to solve for $f_3{}^c$ when $y \geq 0$ and $z \geq 0$. We reflect the flow symmetrically across the planes $y = 0$ and $z = 0$ into the other quadrants and then our solution is given by the source distribution

$$f_3{}^c = -\frac{1}{\pi}\iint_{A_1+A_2} \frac{1}{R_B}\frac{\partial f_3{}^c}{\partial \eta}(\xi,0^+,\zeta)d\xi\, d\zeta - \frac{1}{\pi}\iint_{A_3+A_4} \frac{1}{R_B}\frac{\partial f_3{}^c}{\partial \zeta}(\xi,\eta,0^+)d\xi\, d\eta \qquad (23)$$

where $R_B \equiv \sqrt{(x-\xi)^2 - B^2(y-\eta)^2 - B^2(z-\zeta)^2}$, and $(f_3{}^c)_\eta$ and $(f_3{}^c)_\zeta$ are given by Eqs. (21) and (22) for $\eta,\zeta \geq 0$; symmetry properties determine the values in the other quadrants. The surface areas $A_1 + A_2$ and $A_3 + A_4$ are the regions of the cruciform reference surface which are enclosed by both the vertex-centered free stream Mach cone and the fore-cone from the field point $P(x,y,z)$. Evidently $f_3{}^c$ vanishes in region IV. This now completes the ordinary second-order solution for cruciform surfaces.

The above equations and solution define a pseudolinearized flow problem, in terms of the potential $f_3{}^c$, which is mathematically the same as the problem that would occur if the linearized theory were used to solve the problem of certain symmetric cruciform wings with sonic leading edges. This analogy allows us to exploit both physical and mathematical aspects of the linearized flow in discussing the properties of $f_3{}^c$. If, for example, wing 1 has an initial positive discontinuity in slope, we see from Eq. (22) that wing 2 in the pseudolinearized problem will have a positive discontinuity in slope along its sonic leading edge. This is known to lead to edge stagnation lines with square-root singularities in the x- and y-components of velocity. Thus $f_{3x}{}^c$ and $f_{3y}{}^c$ will be positively infinite along the line $x - By = 0$. From Eq. (19), we see that the particular integral $f_3{}^p$ will then also introduce a singularity, in the form of a discontinuity in potential, across the surface $x - By = 0$; this can be interpreted to imply Dirac delta-function singularities in the velocity.

Thus many of the interesting features appear when the wings exhibit a discontinuity in slope. The simplest such arrangement occurs for the wedge profiles $Y(x) = xH(x)$ and $Z(x) = xH(x)$, where $H(x)$ is the Heaviside step function. The complete solution for this case may be written

$$\left.\begin{aligned}
\varphi_1 &= -(\tau_1/B)(x - By)H(x - By)\\
\varphi_2 &= -(\tau_2/B)(x - Bz)H(x - Bz)\\
f_1 &= [(\tau_1{}^2/B^2)(x - By) - M^2N\tau_1{}^2x/2B^2]H(x - By)\\
f_2 &= [(\tau_2{}^2/B^2)(x - Bz) - M^2N\tau_2{}^2x/2B^2]H(x - Bz)\\
f_3{}^P &= -[M^2(N-1)/B^2]\tau_1\tau_2[2x - B(y + z)]H(x - By)H(x - Bz)
\end{aligned}\right\} \qquad (24)$$

Using the above values for φ_1 and φ_2 in Eqs. (21), (22), and (23), we have in III

$$f_3{}^c = \frac{\tau_1\tau_2}{B\pi}[1 + M^2(N-1)] \times \left\{\frac{4}{B}\sqrt{x^2 - B^2(y^2+z^2)} + 2z\tan^{-1}\frac{Bz}{\sqrt{x^2 - B^2(y^2+z^2)}} - z\pi + 2y\tan^{-1}\frac{By}{\sqrt{x^2 - B^2(y^2+z^2)}} - y\pi\right\} \qquad (25)$$

with $f_3{}^c$ in $IV \equiv 0$. Differentiation of Eq. (25) shows the square-root singularities in the velocity.

Uniformly Valid Solution and Wave Structure for Wedge Profiles

Removal of the discontinuities in the potential and square-root singularities in the velocities is accomplished by the techniques of Refs. 3 and 4 which involve shifting the flow field relative to limit surfaces and then determining the shock surfaces which are inserted between these limit surfaces. Since we already have at our disposal enough information from our solution to calculate the shock surfaces to a first approximation, we do not have to find the limit surfaces as an intermediate step. At shock surfaces the potential must be continuous and the jump in the normal component across the shock must satisfy Eq. (3). Using the criterion of continuity of potential, we require the second-order solution in order to find a first approximation to the shock position. By expanding Eq. (3) and neglecting $0(\tau^2)$ terms and calculating the surface on which Eq. (3) is satisfied, we require only the velocities as given by linearized theory to find the shock position. These alternate criteria are found in this case to give the same shock surfaces but no generality should be attached to these statements in connection with other types of quasi-cylindrical problems.

Since we already have the ordinary second-order theory it is then simpler to use the criterion of continuity of potential to find the shock surfaces S_1 and S_2 in Fig. 2. Once the shock surfaces have been found, we make the first-order solution uniformly valid by simply considering the velocities as given by the linearized theory to be shifted to the shock surfaces S_1 and S_2. For wedge profiles, the linearized theory gives constant velocities and there is no difficulty in shifting these values. If the profiles were not wedges, this would have to be done by characteristic arguments analogous to Friedrich's[3] or Whitham's theory,[3] or by using the formal mathematics of Lighthill[4] mentioned in the introduction. With the first-order solution made uniformly valid, we then recalculate the second-order solution over the extended region bounded by shock surfaces S_1 and S_2. This for wedge profiles gives the same differential equation and boundary conditions Eqs. (7), (10), and (11) but over the ex-

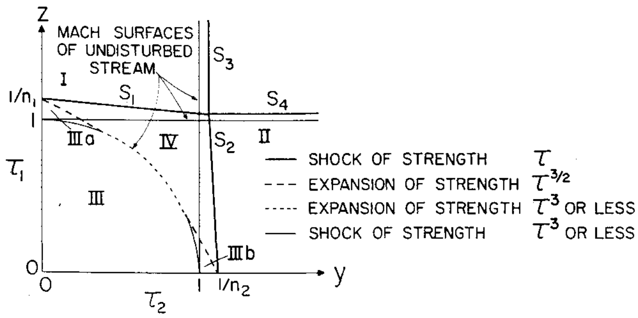

FIG. 2. Shock structure in plane of constant x.

tended regions III and IV for the second-order potential f. The same decomposition of the second-order potential given by Eqs. (17) and (19) is valid, but now f_1, f_2, and f_3^p are continued forward to the shock surfaces S_1 and S_2. A change in f_3^c occurs which is significant but f_3^c as calculated will have no effect near S_1 and S_2. This result might be anticipated by analogy of the problem in f_3^c with the linearized theory of conical flow. It is the above facts, applicable only when the profiles are wedges, that allow us to use the criterion of continuity of potential to find the shock surfaces S_1 and S_2; then

$$[\varphi + f] = 0 \quad \text{gives } \varphi_2 + f_2 + f_3^p = 0 \quad \text{on } S_1 \tag{26}$$

$$[\varphi + f] = 0 \quad \text{give } \varphi_1 + f_1 + f_3^p = 0 \quad \text{on } S_2 \tag{27}$$

which gives for S_1

$$x - By\, M^2(N-1)\tau_1/B - Bz(1 - M^2N\tau_2/2B - M^2(N-1)\tau_1/B) = 0 \tag{28}$$

and for S_2

$$x - By(1 - M^2(N-1)\tau_2/B - M^2N\tau_1/2B) - Bz\, M^2(N-1)\tau_2/B = 0 \tag{29}$$

The boundary conditions for f_3^c, Eq. (21) and (22), remain the same but the limits of integration in Eq. (23) now extend forward to S_1 and S_2. This removes the stagnation point singularity in the derivatives of $-f_3^c$ and it now refers to two intersecting delta wings of constant slope with slightly supersonic leading edges. The sweep parameters n_1 and n_2 where $n \equiv k/B$ and k is the tangent of the sweepback angle, are given by

$$1/n_1 = 1 + M^2N\tau_2/2B + M^2(N-1)\tau_1/B \tag{30}$$

$$1/n_2 = 1 + M^2N\tau_1/2B + M^2(N-1)\tau_2/B \tag{31}$$

The solution for f_3^c is then

In III

$$f_3^c = \frac{\tau_1\tau_2}{B\pi}\,[1 + M^2(N-1)]\left\{\frac{x - Bn_2y}{B\sqrt{1-n_2^2}}\cos^{-1}\frac{(xn_2 - By)}{\sqrt{(x - Bn_2y)^2 - B^2z^2(1-n_2^2)}} + \frac{(x + Bn_2y)}{B\sqrt{1-n_2^2}} \times \cos^{-1}\frac{(xn_2 + By)}{\sqrt{(x + Bn_2y)^2 - B^2z^2(1-n_2^2)}} + \frac{(x - Bn_1z)}{B\sqrt{1-n_1^2}}\cos^{-1}\frac{(xn_1 - Bz)}{\sqrt{(x - Bn_1z)^2 - B^2y^2(1-n_1^2)}} + \frac{(x + Bn_1z)}{B\sqrt{1-n_1^2}}\cos^{-1}\frac{(xn_1 + Bz)}{\sqrt{(x + Bn_1z)^2 - B^2y^2(1-n_1^2)}} - z\left[\pi + \tan^{-1}\frac{xy - Bn_2(y^2+z^2)}{z\sqrt{x^2 - B^2(y^2+z^2)}} - \tan^{-1}\frac{xy + Bn_2(y^2+z^2)}{z\sqrt{x^2 - B^2(y^2+z^2)}}\right] - y\left[\pi + \tan^{-1}\frac{xz - Bn_1(y^2+z^2)}{y\sqrt{x^2 - B^2(y^2+z^2)}} - \tan^{-1}\frac{xz + Bn_1(y^2+z^2)}{y\sqrt{x^2 - B^2(y^2+z^2)}}\right]\right\} \tag{32}$$

In IIIa

$$f_3^c = \frac{\tau_1\tau_2(1 + M^2(N-1))}{B^2\sqrt{1-n_1^2}} \times \left\{x - Bn_1z - By\sqrt{1-n_1^2}\right\}$$

In IIIb

$$f_3^c = \frac{\tau_1\tau_2(1 + M^2(N-1))}{B^2\sqrt{1-n_2^2}} \times \left\{x - Bn_2y - Bz\sqrt{1-n_2^2}\right\}$$

In IV

$$f_3^c = 0$$

The second-order solution for wedge profiles that is uniformly valid to first order now consists of Eq. (24) continued forward to the shock surfaces S_1 and S_2 and f_3^c from the preceding paragraph. The solution now has no discontinuities in potential or singular velocities. If desired we can find the second-approximation to the shock surfaces S_1 and S_2 by retaining quadratic terms in Eq. (3) and solving for the $0(\tau^2)$ corrections to these surfaces using the velocities given by the second-order theory. In Eq. (32), $1/\sqrt{1-n^2} \backsimeq 1/\sqrt{2(1-n)}$ is of $0(\tau^{-1/2})$ and the expansion of the solution $\varphi + f$ adjacent to the intersection of the Mach cone with the reference surfaces contains terms of $0(\tau)$, $0(\tau^{3/2})$, $0(\tau^2)$, and $0(\tau^{5/2})$. The terms of $0(\tau^{5/2})$ must be neglected. Away from the Mach cone the expansion of the uniformly valid solution is identical with the expansion of the ordinary solution $\varphi + f$ given by Eqs. (24) and (25); these contain terms of $0(\tau)$ and $0(\tau^2)$. The reason for this is that the inverse cosines in Eq. (32) are expanded about an argument of (-1) near the Mach cone and about $(+1)$ away from it. The result means that, in the interior of the Mach cone, the ordinary second-order solution is also identical with the second-order solution that is uniformly valid to second-order.

The wave structure in the plane $x = 1$ for $y \geq 0$, $z \geq 0$ is shown in Fig. 2. The shock surfaces S_3 and S_4 have been determined previously in reference 5 as

$$S_3\!: \quad By = x(1 + M^2N\tau_1/2B) \tag{33}$$

$$S_4\!: \quad Bz = x(1 + M^2N\tau_2/2B) \tag{34}$$

Altogether, the wave system in the first quadrant consists of four planar shocks, S_1, S_2, S_3, and S_4, with strengths of order τ, two oblique Prandtl-Meyer fans with strengths of order $\tau^{3/2}$, an expansion wave with strength of order τ^3 or less comprising a segment of the undisturbed Mach cone and two weak shocks with strength τ^3 or less comprising the remaining portion of the Mach cone. These last two statements are based upon the results of Ref. 4 where it is shown that the shock strength near the Mach cone for a planar wing with supersonic leading edges in linearized theory is $0(\alpha^2)$. Since we have the analogy between our solution $f_3{}^c$ and the linearized theory, $\alpha^2 \sim \tau^3$.

The surface pressure is given by

$$c_p = -2(\varphi_x + f_x) - \varphi_y{}^2 - \varphi_z{}^2 + B^2\varphi_x{}^2 \equiv c_{p_1} + c_{p_2} + c_{p_3} \quad (35)$$

where the right side is evaluated on the reference surface, c_{p_1} refers to wing 1 alone, c_{p_2} refers to wing 2 alone and

$$c_{p_3} = 2B^2\varphi_{1x}\varphi_{2x} - 2f_{3x}{}^c + 2M^2(N-1)(\varphi_1\varphi_2)_{xx} \quad (36)$$

The proper method of calculating the surface pressures is to use the ordinary second-order theory away from the vertex-centered Mach cone and the uniformly valid theory near and ahead of the Mach cone. The reason is that the uniformly valid solution away from the Mach cone contains unnecessary higher-order terms. For $M = 3$, $\gamma = 1.405$, and $\tau_1 = 5°$, Fig. 3 shows the variation of c_p with the conical variable $\delta \equiv By/x$ along the surface of wedge 2 for the cases $\tau_2 = 3°$ and $\tau_2 = 0$, the latter two-dimensional case being provided for reference. Depicted are the two second-order solutions and the two linearized solutions. The position of the free-stream Mach cone is given by $\delta = 1$, the "difficult" point. The two second-order solutions are identical for $\delta < 0.8$; the near constancy of c_p in this range occurs because the rapidly varying contributions of each cruciform source distribution generating $f_3{}^c$ happen virtually to balance each other here. For $0.8 < \delta < 1.25$, the predictions of ordinary second-order theory are entirely spurious. It is clearly of some practical importance to modify its predictions by adjusting the wave structure with the singularities as the guide.

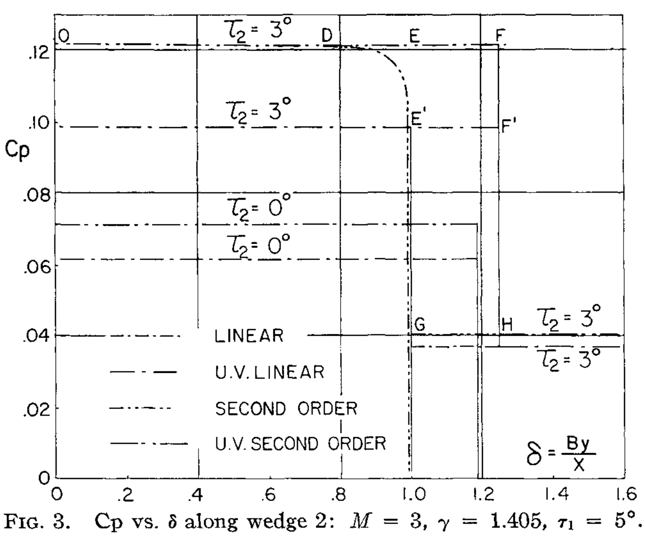

FIG. 3. Cp vs. δ along wedge 2: $M = 3$, $\gamma = 1.405$, $\tau_1 = 5°$.

But it is not always necessary in practical situations actually to obtain the uniformly valid solution, once the defects in the ordinary second-order solution are appreciated. Consider, for example, the aerodynamic forces implied by Fig. 3 and suppose that a uniformly valid first-order or linearized solution is at hand. Then the abscissa of point F is known, the segment $E'F'$ is known, and the only unknown portion of the loading is $DEFF'E'$. This area is $0(\tau^{5/2})$ and negligible compared with the gross area, which is $0(\tau)$. Its consideration is then necessary only to the extent that it removes the spurious singularity at E. On the other hand, $E'F'HG$ is $0(\tau^2)$ and always necessary. A further $0(\tau^2)$ shift in the wave FH, such as is given by a second-order solution that is uniformly valid to second order, leads to an area which is $0(\tau^3)$ and consequently negligible in c_p.

Uniformly Valid Solution for Arbitrary Cruciform Profiles and Other Quasi-Cylinders

The difference between the wave front structure for wedge profiles and arbitrary profiles with discontinuities in slope consists in an attenuation of $0(\tau^2)$ of the wave front as we proceed away from the body. What remains to be done is to find how to adjust the ordinary second-order solution for arbitrary cruciform surfaces and how to find the shock surfaces S_1 and S_2. To do this we use the general formalism of Ref. 4. These equations are as follows:

$$\Phi = \varphi(u,y,z) + f(u,y,z) + 0(\tau^3);$$

$$\varphi = 0(\tau), \quad f = 0(\tau^2) \quad (37)$$

$$x = u + \tau x_1(u,y,z) + \tau^2 x_2(u,y,z) + 0(\tau^3) \quad (38)$$

$$-B^2\varphi_{uu} + \varphi_{yy} + \varphi_{zz} = 0 \quad (39)$$

$$-B^2 f_{uu} + f_{yy} + f_{zz} = 2\{[M^2(N-1)B^2\varphi_u - B^2 x_{1u}]\varphi_{uu} + (M^2\varphi_y + x_{1y})\varphi_{uy} + (M^2\varphi_z + x_{1z})\varphi_{uz}\} - \varphi_u(B^2 x_{1uu} - x_{1yy} - x_{1zz}) \quad (40)$$

When x_1 is determined by requiring the second-order solution to exhibit the correct behavior, we then have a second-order solution uniformly valid to first order. The determination of x_2, by requiring the third-order solution to have correct behavior, makes the solution uniformly valid to second-order. The boundary conditions transform in a similar manner.

These equations reduce to the previous equations when x_1, x_2, etc. are suppressed. A particular integral f_{00} which completely accounts for these terms in Eq. (40) is given in Ref. 6. Letting $f = f_{00} + f_0$, we have

$$\Phi(u,y,z) = \varphi(u,y,z) + f_0(u,y,z) + \tau x_1(u,y,z)\varphi_u(u,y,z) + 0(\tau^3) \quad (41)$$

The differential equations for φ and f are now formally the same as in the ordinary theory with x replaced by u. Near the singular surfaces, the inversion of Eq.

(38), giving $u(x,y,z;\ \tau)$ is not single-valued and certainly does not have a unique power series representation in τ; this is in fact the crux of the method. We can expect, however, that regions exist away from the singular surfaces for which

$$u = x - \tau u_1(x,y,z) - \tau^2 u_2(x,y,z) + 0(\tau^3) \quad (42)$$

is valid. Substitution of Eq. (42) in Eq. (41) or Eq. (36) implies at once away from the singular surface the ordinary expansion of Φ. In terms of Eq. (41), the result is

$$\Phi = \varphi(x - \tau u_1 - \ldots, y,z) + f_0(x - \tau u_1 - \ldots, y,z) + \tau x_1(x - \tau u_1 - \ldots, y,z)\tau_u(u = x - \tau u_1 - \ldots, y,z) = \varphi(x,y,z) + f(x,y,z) + 0(\tau^3) \quad (43)$$

Eq. (43) permits identification of the function φ and f with their counterparts in ordinary second-order theory. If φ and f are determined by the latter theory, then Eq. (43) results, and the uniformly valid solution [Eq. (41)] can be constructed by replacing x by u and adding $x_1(u,y,z)\varphi_u(u,y,z)$ to the ordinary second-order theory. Of course, Eq. (38), giving u, is still unknown. This result represents a continuation of the ordinary second-order solution to the neighborhood of the singular surfaces. Eq. (41) is most useful when only x_1 but not x_2 is to be determined, for it then contains in principle the means for finding x_1.

For cruciform surfaces, the ordinary second-order solution for arbitrary cross-sections given by Eqs. (12), (13), (15), (16), and (23) is made uniformly valid to first order by application of the above method. We determine x_1 by requiring the potential to be continuous at the shock surfaces. This gives for arbitrary cruciform surfaces the same $0(\tau)$ term for the position of the shock surfaces S_1 and S_2 as that for the case of wedge profiles and at the same time justifies the procedure used to find the solution for wedge profiles.

Eq. (41) is valid for other quasi-cylindrical problems in which x_1 might be determined using criteria other than continuity of potential. It may, for instance, be used to remove any singularities in the velocity components that can occur by using the particular integrals that are available in second-order theory. The second-approximation to the position of the shock surfaces requires a knowledge of x_2 which is implied by the third-order problem. In general x_2, as shown in Ref. 4, is multi-valued, giving two sets of values for the function $x(u,y,z;\ \tau)$ but the regions of validity of these two sets of values overlap and it is in this region of overlap that a shock must be inserted so as to satisfy the appropriate shock conditions. Having completed discussions of the cruciform problem, we now give an extended version of Lighthill's rule (stated in the introduction), which is then also valid for quasi-cylindrical bodies: The second-order flow field is made uniformly valid to second-order if distance downstream from the undisturbed nose Mach cone is always reinterpreted therein as distance downstream from the limit cone if in addition the effect $x_1(x,y,z)\text{-}\varphi_x(x,y,z)$ of coordinate distortion is first added to the ordinary second-order potential. The function $x_1(x,y;z)$ has to be determined to insure that the second-order solution exhibits the correct behavior near singular surfaces. The shock surface is then inserted between the undisturbed Mach cone and the limit cone.

Concluding Remarks

The solution and wave structure when τ_1 and τ_2 are negative in the first quadrant are not given by the previous solution without further modification. The reason for this is that the first-order theory allows expansion shocks and the requirement that the second-order potential be continuous at these expansion shocks will not remove the singularities in f_3^c in the second-order solution. The singularities that occur are formally the same as that occur for subsonic leading edges in linearized theory. This failure is of course is desirable since expansion shocks are physically unreasonable and the mathematics of the iteration procedures shows this. For the case $\tau_1,\tau_2 < 0$ in the first quadrant, the ordinary second-order solution given in this paper can be made uniformly valid by using the actual Prandtl-Meyer fans that occur in the solution for the isolated wings rather than expansion shocks. This can be construed as a smoothing procedure to remove the singularities in the velocities. With these modifications all possible cruciform surfaces of high aspect ratio can be solved under the assumptions of second-order theory.

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